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NON-UNIFORM TORSION OF PLATE GIRDERS

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SYNOPSIS

As one part of a comprehensive investigation of the torsional behavior of built-up structural members, a group of full-size plate girder specimens were tested under non-uniform torsion conditions. The measured stresses and angular distortions were compared with calculated values based on available theories. It was found that the Timoshenko theory checked out well for the rolled beam and for the plate girders with relatively high ratio of web to flange thickness. However, it was inadequate when applied to specimens with a low ratio of web to flange thickness in which the web deformation effect becomes significant.

A procedure based on the compatibility of test data, by which measured web and flange distortions were used to compute the flange bending stresses, established the validity of a suggested grouping of section properties for plate girders. Subsequent application of the Goodier-Barton theory, which considers the web deformation effect, indicated that the method can be successfully adapted to plate girders. The effect of a reduction in the torsion constant under non-uniform torsion loading was also investigated.

INTRODUCTION

The behavior of solid sections under uniform and non-uniform torsion has been studied by many investigators. Most of the theories developed for rolled beams have been checked experimentally, with the notable exception of the effect of web deformation on beams subjected to non-uniform torsion. The whole field of built-up members under torsion has not been systematically investigated until recently.

A comprehensive study of full-size plate girders under torsion loading was undertaken at the Fritz Engineering Laboratory of Lehigh University. A special torsion testing machine had been designed and built for this purpose.

The first phase of the program, covering the uniform torsion of built-up structural members has been completed by Chang and Johnston.[4] The third

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phase will be based on the combined bending and torsion tests performed at Swarthmore College with the cooperation of Professor Samuel T. Carpenter.

This report, which deals with the non-uniform torsion problem of plate girders, completes the second phase. The main objectives were to compare the theoretically-predicted with observed behavior and to determine the modifications which were indicated by the characteristics of the built-up section.

Summary of Torsion Theory

Classification of Torsion

The development of the torsion theory for structural members is well documented. Following the pioneer work of St. Venant,[5] major contributions have been made by Prandtl,[6] Timoshenko[7,8] and many others. Only the pertinent highlights of the available background material will be reviewed here in order to form a frame of reference for this present investigation.

A twisted structural member is considered to be in a state of uniform torsion when the angle of twist per unit length is constant over portions subjected to constant torque. Since no significant tendency to warp longitudinally exists in circular sections, these sections will always be in uniform torsion regardless of the condition of end restraint. The terms pure, simple, and St. Venant torsion have also been used to designate this condition in which only transverse torsional shear stresses exist over the cross-section.

Non-circular sections when twisted undergo longitudinal warping as well as angular distortions. Plane sections before twisting no longer remain plane after twisting. If this warping tendency is not restrained, the member will still exhibit a constant unit angle of twist and will therefore be subjected to uniform torsion. If, however, the warping distortion is arrested in any way, longitudinal normal stresses are produced in addition to the torsional shear stresses. The unit angle of twist will no longer be constant along the length. Therefore, non-uniform torsion is considered to exist in a non-circular section with restraint of warping.

The behavior under torsion of open sections such as I-beams and plate girders is significantly altered by certain support and loading conditions which introduce longitudinal restraint. Some common cases of non-uniform torsion loading with and without vertical bending are the following:

1. Cantilevered beam under shaft torque loading.
2. Cantilevered beam with eccentrically applied transverse load.
3. Simply supported beam subjected to a twisting couple at some intermediate point.
4. Simply supported beam with an eccentrically applied transverse load.

Uniform Torsion Theory

The relationship between the applied torque (T) and the angular distortion for members under uniform torsion is expressed by

$$T = KG\frac{d\psi}{dz} = KG\theta \qquad (1)$$

where K is the torsion constant, (in.^{4})

G is the modulus of elasticity in shear, (p.s.i.)
ψ is the angle of twist, (rad.)
z is the variable distance along the longitudinal axis, (in.)
θ is angle of twist per unit length, (rad. per in.)

The torsion constant is a function of the geometry of the cross-section and can be calculated or evaluated experimentally. The value for a narrow rectangle, as developed by St. Venant, is

$$K = \tfrac{1}{3} w t^3 - 0.21 t^4 \qquad (2)$$

where (w) is the width and (t) is the thickness.

A wide flange beam with parallel-sided flanges can be considered as an aggregation of three component rectangles. The end loss effect, represented by the last term of Eq. (2), can be neglected in design calculations since it is small and offset by other compensating factors.

The torsion constant can be evaluated experimentally for solid sections by use of the membrane analogy proposed by Prandtl.[6] This technique has been used by Taylor and Griffith,[9] Trayer and March,[10] and by Lyse and Johnston[11] to check the torsion constant for a wide range of structural sections.

Another laboratory procedure to evaluate (K) is to subject the member to a shaft torque under uniform torsion conditions. The torque is plotted against the unit angle of twist. From equation (1) the torsional rigidity KG will be equivalent to the slope of the straightline portion of the curve.

The uniform torsion problem for built-up structural members, held together by bolts, rivets or welds, has been investigated by Madsen[12] and by Chang and Johnston.[4] A typical plate girder section would have a torsional rigidity intermediate between the extreme limits of separate-action and solid-section behavior—a considerable difference in most cases.

At the upper limit the cross-section would be equivalent to that of a rolled beam of the same external dimensions. The lower limit, separate-action K_S is computed on the premise that all elements of the cross-section twist through the same angle but independently. Relative warping distortions are not restrained by the connectors. For a plate girder dimensioned as shown in Fig. (1-a) with (n) cover plates of equal thickness per flange the torsion constant K_S is given by

$$K_S = \tfrac{1}{3}\left[2nwt_F^3 + ht_W^3 + 4\left(e + f + m + \tfrac{b - t_W}{2}\right)t_A^3\right] \qquad (3)$$

Since the elements of a bolted or riveted plate girder are actually connected along well-defined gage lines and interplane friction does exist, the so-called integral action constant K_I must fall somewhere between the two extreme values. Based on the concept proposed de Vries,[13] it is assumed that the core portion in each flange acts as a solid section while the remainder of the section acts as separate rectangles when the section is twisted. The core is taken to the Tee-section formed between the outside pair of gage lines in the flange and the outside gage line in the downstanding leg of the flange angle as shown in Fig. (1-b).

As the integral action constant for use in design, Chang and Johnston proposed

$$K_I = \tfrac{2}{3} b T_F^3 + \tfrac{1}{3} e T_W^3 + \tfrac{4}{3} nct_F^3 + \tfrac{4}{3} mt_A^3 + \tfrac{4}{3} ft_A^3 + \tfrac{1}{3} at_W^3 \qquad (4)$$

This value is the effective value of the torsion constant if longitudinal continuity is insured. The longitudianl pitch (p) of the connectors in both legs of

the flanges angles should not be greater than the critical pitch (p')

$$P' = A + T \tag{5}$$

where A = rivet or bolt head diameter.
T = overall thickness of connected plates, all dimensions in inch units.

If the actual longitudinal rivet pitch exceeds the critical pitch in either the horizontal or vertical leg of the flange angle, Chang and Johnston have proposed a reduced effective torsion constant which can be computed from the K_S, K_I and p' values.

Experimental verification of the proposed formulations of the torsion constant for a wide range of built-up members has been obtained at the Fritz Engineering Laboratory of Lehigh University. A preliminary series of tests on small-scale model specimens performed by Madsen indicated that actual torsion constant was considerably smaller than the corresponding solid section value. Later a representative group of full-size plate girder specimens fabricated under controlled shop conditions was subjected to uniform torsion loading by Chang and Johnston. They found that there was reasonably good agreement between the value computed from the proposed design formulas and that obtained experimentally.

In a supplementary series of tests on built-up column sections, Jentoft, Mayo & E. R. Johnston[14] were able to check quite closely the design formulas for effective K which have been proposed for members without longitudinal continuity.

Non-uniform Torsion Theory - Timoshenko Solution

The basic relationships governing the non-uniform torsion behavior of bi-symmetrical open sections such as I-beams and plate girders can be illustrated by a beam fixed at one end and subjected to a shaft torque at the other as in Fig. (2).

If the flanges were separated from the web and a torque applied as a couple in the form of two equal shear forces acting in opposite directions at the flange centroids, the flanges would deflect laterally as two cantilevers without twisting. The torque at any section would then be equal to the flange shear V_F times the distance (h) between flange centroids. This couple will be designated the restraint of warping torque T_W since the lateral bending of the flange exists by virtue of the condition of restraint at one end. This factor has also been called the flange bending or torsion bending torque.

Since the flanges are in reality connected to the web, it is assumed in the Timoshenko theory that the elements twist through the same angle. Therefore, part of the applied torque is resisted by the torsional rigidity of the whole section, the rest by the restraint of warping torque. The torsional shear stresses combine to form the torsional shear torque T_S which is given by Eq. (1) for the uniform torsion case.

The relationship between the angle of twist and the lateral bending of the flange is based on the geometry of the twisted section. Using the notation shown in Fig. (2), the lateral displacement of the flange centroid can be written for small angles of twist ψ as

$$X = \frac{h}{2} \sin \psi \approx \frac{h}{2} \psi \tag{6}$$

After the elementary beam theory and certain substitutions are introduced, the equilibrium equation of torsion can be written as

$$T = T_S + T_W = KG\frac{d\psi}{dz} - \frac{EI_y h^2}{4}\frac{d^3\psi}{dz} \qquad (7)$$

where KG is the torsional rigidity and EI_y is the flexural rigidity about the vertical axis.

The general solution together with the assumed boundary conditions lead to the following expressions for the angular distortions:

$$\psi = \frac{T}{KGr}\left[rz - \sinh rz + \tanh rL(\cosh rz - 1)\right] \qquad (8)$$

$$\theta = \frac{d\psi}{dz} = \frac{T}{KG}\left[1 - \frac{\cosh r(L-z)}{\cosh rL}\right] \qquad (9)$$

$$\text{where} \quad r = \frac{1}{a} \qquad a = \frac{h}{2}\sqrt{\frac{EI_y}{KG}} \qquad (10)$$

The flange bending moment, from which the bending stresses can be computed, is expressed by

$$M_F = \frac{T}{rh}\left[\frac{\sinh r(L-z)}{\cosh rL}\right] \qquad (11)$$

The web shear stress is the torsional shear stress at the point. The flange shear stress at any point is obtained by superimposing the corresponding transverse and torsional shear stresses. The variation of representative functions along the span for the cantilevered I-beam under non-uniform torsion loading is summarized in Fig. (3).

The uniform torsion solution is also presented for comparison. The net effect of the restraint of warping is to reduce the angular distortions while increasing the longitudinal stresses in the flange.

Goodier-Barton Theory

It has been pointed out by Goodier and Barton[15] that the Timoshenko solution has certain limitations inherent in its derivation. For example, the division of torque at the free end cannot be modified to fit different boundary conditions. Certain of these limitations are removed by the Goodier-Barton theory which considers the effect of web deformation on the torsional behavior of I-beams.

The twisting of an I-beam under non-uniform torsion loading is geometrically related to the lateral deflection of the flange. Such deflection implies a non-zero fourth derivative corresponding to some distributed load on the flange which must be supplied by the web. The interacting forces between web and flange—lateral load (q) per unit length and twisting moment (m) per unit length—cause the web to deform as a plate into an S-curve. This web deformation, shown highly magnified in Fig. (4) is accompanied by a reduction in the flange angle of twist by an angle (α) equal to the web chord rotation.

It is demonstrated that the relative dimensions of web and flange influence the quantitative division of applied torque between restraint of warping and torsional shear torques.

By careful consideration of the lateral bending and twisting of the flange, the bending and twisting of the web as a plate, Goodier and Barton have deduced a set of simultaneous differential equations in terms of the angular distortions (ψ) and (α).

$$k_3 h^2(\psi'' - \alpha'') + k_4 \alpha = 0 \tag{12}$$

$$k_1 h^4 \psi^{IV} - k_2 h^2 \psi'' + 2k_4 \alpha = 0 \tag{13}$$

$$k_1 = \frac{EI_y}{4KG}; \quad k_2 = \frac{K_W}{K}; \quad k_3 = \frac{K_F}{K}$$

$$k_4 = \frac{6D_W h}{KG}; \quad D_W = \frac{E t_W^3}{12(1-\mu^2)}; \quad \mu = \text{POISSON'S RATIO} \tag{14}$$

K_W = Torsion constant for web

K_F = Torsion constant for flange

By substituting the exponential forms

$$\psi = C_1 e^{\lambda z} \qquad \alpha = C_2 e^{\lambda z} \tag{15}$$

in the system of simultaneous differential equations, and setting the determinant of the coefficients of C_1 and C_2 equal to zero, a quadratic equation in $(\lambda h)^2$ is obtained whose roots are given by

$$(\lambda h)^2 = \frac{K_1}{2}\left[1 \pm \sqrt{1 - K_2}\right] \tag{16}$$

$$K_1 = \frac{k_2}{k_1} + \frac{k_4}{k_3} \qquad K_2 = \frac{\frac{4k_4}{k_1}\left(2 + \frac{k_2}{k_3}\right)}{K_1^2} \tag{17}$$

For any given specimen the combined dimensionless constants K_1 and K_2 and the two roots $(\lambda h)^2$ can be computed from the dimensions and properties of the section. The roots are real or complex depending on whether K_2 is less than or greater than unity. I-beam sections with relatively thick webs have been found to yield real roots.

When the roots of the quadratic are real, there will be four possible values of λ, designated $\pm \lambda_1$ and $\pm \lambda_2$.

The complete solution for the angle of twist and the flange reduction is given as

$$\psi = A_1 + A_2 z + A_3 e^{\lambda_1 z} + A_4 e^{-\lambda_1 z} + A_5 e^{\lambda_2 z} + A_6 e^{-\lambda_2 z} \qquad (18)$$

$$\alpha = [k_5(\lambda_1 h)^2 - k_6(\lambda_1 h)^4][A_3 e^{\lambda_1 z} + A_4 e^{-\lambda_1 z}]$$
$$+ [k_5(\lambda_2 h)^2 - k_6(\lambda_2 h)^4][A_5 e^{\lambda_2 z} + A_6 e^{-\lambda_2 z}] \qquad (19)$$

$$k_5 = \frac{k_2}{2k_4} \qquad k_6 = \frac{k_1}{2k_4} \qquad (20)$$

The arbitrary constants are evaluated by introducing a system of compatible boundary conditions. The "long beam" solution is based on an assumed condition at the free end which is not attained by the relatively short specimens used in the present investigation.

In the "finite beam" solution the specimen is assumed to be rigidly attached at both ends to heavy plates which remain parallel to each other and normal to the longitudinal axis while twisting by equal amounts in opposite directions as shown in Fig. (5).

With the origin at the center of the span, the angular distortions for this special case are expressed by

$$\psi = A_2 z_2 + B_3 \sinh \lambda_1 z_2 + B_5 \sinh \lambda_2 z_2 \qquad (21)$$

$$\alpha = [k_5(\lambda_1 h)^2 - k_6(\lambda_1 h)^4] B_3 \sinh \lambda_1 z_2$$
$$+ [k_5(\lambda_2 h)^2 - k_6(\lambda_2 h)^4] B_5 \sinh \lambda_2 z_2 \qquad (22)$$

$$B.C. \quad at\ z_2 = L, \quad \alpha = 0; \ \psi' = 0; \ \psi = \psi_m$$

From the boundary conditions at the fixed end, the arbitrary constants A_2, B_3 and B_5 can be evaluated in terms of max. angle of twist ψ_m.

The equilibrium equation of torsion considering web deformation is given as

$$T = KG\psi' - \frac{1}{4} E I_y h^2 \psi''' - 2K_F G\alpha' \qquad (23)$$

This equation is used to find ψ_m in terms of the applied torque. When this value is substituted back, the desired angular distortions and stress functions can be evaluated.

Experimental Program

Test Specimens

An extensive investigation of the behavior of full-size built-up structural members under uniform torsion loading was underway at the Fritz Engineering

Laboratory in the summer of 1949. Since the initial tests were kept in the elastic range, it was felt that the same specimens could be used in the second series of tests designed to study the behavior of plate girders under non-uniform torsion. Seven specimens were made available for this purpose.

The important features and section properties of the specimens tested are summarized in Table (1). Design and fabrication details are to be found in the uniform torsion report.(4) All specimens were 14'9" in length.

The high-strength bolts used in the bolted specimen were all tightened to a value of 300 ft.-lbs. by means of a calibrated torque wrench to insure integral action in the low load range. All rivets were machine-driven under controlled shop conditions. Welds were continuous except for those welds used to connect the cover plates to flanges.

The experimental value of the torsion constant $(K)_A$, later used in computation "A," was obtained from the uniform torsion tests. The torsion constant $(K_S)_D$ was computed on the assumption of separate action behavior.

Test Set-up

The analytical solutions are predicted on the assumption that one end is completely restrained. This fixed-end condition is extremely difficult to provide externally. The logical arrangement was to attain fixity internally by loading the specimen symmetrically about the center-line of the span.

The specimen was simply supported at both ends by web framing angles attached to the end support plates of the special torsion testing machine. The ends of the specimen were held against twisting by a pair of flange lugs at diametrically opposite corners.

The torsional moment was introduced at the center of the span by a specially designed yoke with cantilevered loading arms. The yoke, when placed around the specimen, was held in position by a series of special fillers each backed by a matching pair of machined wedges. A schematic sketch of the test set-up is shown in Fig. (6).

Disk weights were suspended directly from the end of one loading arm. An equal-force was applied in the opposite direction at the other end by means of suspended weights attached to the loading arm by a cable passing over a ball-bearing pulley.

Under this arrangement a specimen was divided into two symmetrical half-spans each subjected to the same shaft torque. No vertical bending loads were introduced. Complete restraint against longitudinal warping was automatically provided at the plane of symmetry for each half. Except for some unavoidable contact friction at the flange lugs, the outer ends were free to warp.

Instrumentation

Since each specimen was divided into two equal halves, strains were measured in one half and angular distortions in the other as pictured in Fig. (7).

In order to obtain a check on the bending and shear stress distribution which varies along the span as well as around the section, an extensive network of strain gages was required. Approximately 70 SR-4 strain gages were cemented to each specimen at selected stations.

A 3-inch level bar equipped with a spirit level and an adjustable micrometer screw was used to measure the angle of twist. The portable level bar could be seated in prepared holes provided at the top of the level bar supports.

Since the angular distortions are not constant along the span, it was necessary to have an integrated system of angle-measuring stations. The level bar supports for both flange and web were uniformly spaced 11 inches apart along the longitudinal axis of each specimen. For the flange these supports were tack-welded to the top along the longitudinal centerline.

Level bar supports, if attached directly to the web, would have led to erroneous readings because of web deformation. A more representative value of web twist was obtained by basing the angular measurement on the movement of the flange sides. An auxiliary vertical bar, equipped with a hook at the top end, was held tightly against the sides of the flanges by elastic bands. The level bar support was tack-welded to this vertical bar.

Laboratory Tests

Most of the preliminary tests, such as tension tests of coupons, had already been carried out as part of the uniform torsion program. For use in computations the following average values were adopted: $E = 29{,}500$ k.s.i., $G = 11{,}450$ k.s.i.

One plate girder specimen was subjected to a lateral bend test in order to check the computed value of I_y. Since the comparison was favorable, the computed I_y was assigned to all specimens.

After a series of dry runs to shake down the specimen, the shaft torque loading was applied and removed in increments. Observations were recorded at each stage.

The magnitude of the torque applied to each specimen was limited to a value which would insure elastic behavior. In order to have basis for comparison between the various specimens, an arbitrary end torque of 40,000 in.-lb. was established as a standard and used in the subsequent computations.

The test data was first examined by plotting the increment of reading against the corresponding increment of end torque. With the exception of an explainable shift of some readings in the built-up specimens and some irregularity in lightly-stressed gages, the test data showed good linearity.

Wherever possible strain gages were paired so that the average test value could be used. The conversion of SR-4 strain readings to stresses considered the transverse correction.

The level bar readings were converted to angles of twist by the tangent relationship between micrometer movement and gage length. The unit angle of twist for a point midway between two adjacent stations was obtained by dividing the relative change in twist angle by the distance between the stations.

Analytical Study of Results

Comparison of Theories for Rolled Sections

Since the available theories are based on an ideal solid section, they were first applied to Specimen T9, a wide-flange steel beam incorporated in the test program as a control. The solution for the angle of twist was extended to obtain the unit angle of twist as well as the normal and shear stresses in web and flange. However, for comparative purposes only the angle of twist and the flange bending stress variation will be discussed.

The Timoshenko solution, designated Comp. "A," was based on the test value of the torsion constant $(K)_A$. The actual distance from the vertical axis to the line of strain gages in the flange was used in the computation for flange normal stress.

The "finite beam" solution of the Goodier-Barton theory was used in this comparative study. The transposition of the computed values from the origin of coordinates at the torsion-free end to the restrained end was required to make the results comparable. It was also necessary to divide the torsion constant for the section into separate portions for the web and flange. The computed and measured values of the flange bending stress are compared graphically in Fig. (8).

For this specimen the Timoshenko and Goodier-Barton solutions are practically identical, as predicted by the latter. The web deformation effect is small in this 18 WF 77 beam with a relatively thick web. The comparison of both computed curves with measured values is very good.

The angle of twist curves are compared in Fig. (9).

The Timoshenko solution yields one value of angular distortion for the section as a whole while the Goodier-Barton solution gives separate values for the web and flange. The boundary conditions assumed in the "finite beam" solution are such that the flange reduction angle (α) is equal to zero at both ends, as is evident from the converging computed curves for web and flange twists. The angle (α), being the angle by which the flange lags the web, is represented by the difference between the web and flange twist angles. The measured values show a tendency to diverge at the free end. This reflects in part the unavoidable local effect of the flange lugs, which are used to support the section from twisting at the ends. However, at other sections the magnitude of the measured flange reduction angle checks quite well with the anticipated values. In general, the solid sections represented by the rolled beam and the welded girder without cover plates performed as predicted by the Timoshenko and Goodier-Barton theories.

Application of Theories to Plate Girders

In a plate girder some questions exist as to the proper assumptions to make regarding section properties, since the material is not as compactly arranged as in a rolled beam. The effective height h_e, the distance between flange centroids, is one of the dimensions required. It was found by comparison of calculated values that h_e may be varied over a limited range without any significant effect on the results. The usual procedure for locating the centroid therefore may be used, except in the case of a plate girder without cover plates. A reasonable modification, which was tested out on Specimen T1-R, would be to disregard the area represented by the downstanding legs of the flange angles and to consider the center of the remaining portion as the flange centroid.

The Timoshenko solution was applied to the built-up specimens, using the test value of the torsion constant. The angle of twist, the shear stresses, and the flange bending stress curves were compared with experimentally-determined values. The results were favorable for all specimens tested except for T5-R.

The same web thickness of 3/8 in. was used in all the built-up specimens. However, the over-all flange thickness varied from 1/2 in. for T1-R and T3-B, to 1 5/8 in. for T5-R. It was apparent that the ratio of web to flange thickness has an important bearing on non-uniform torsion behavior.

The flange bending stress variation for specimen T5-R computed by the Timoshenko theory is reproduced as curve "A" in Fig. (11). The average discrepancy is about 20% on the low side when compared with measured stresses.

On the basis of these and other observations, it was concluded that the Timoshenko solution has its limitations when applied to certain plate girders. Certain modifications were deemed essential in order to provide an improved prediction of stresses and distortions.

Modification of Theory for Plate Girders

The fact that the actual end supports do not precisely match the theoretically-assumed boundary conditions is a distrubing source of error when computed and measured values are to be compared. One way of by-passing this difficulty would be to base the computations on certain observed values. A

procedure, designated Modification No. 1 and Comp. "H," was devised by which the flange bending stress could be computed from the measured values of the angle of twist.[16]

The basis for this procedure was the fact that the web and flanges do not twist through the same angle. In order to handle the web and the two flanges separately it was necessary to divide the section torsional constant (K) into three parts. As previously developed, the integral action torsion constant (K_I) for a built-up member is the combined effect of the separate action constant (K_S) and solid or core section constant (K_c).

A simple yet logical assumption would be to divide the core portion into two parts by a horizontal line cutting across the downstanding leg of the flange angle as illustrated in Fig. (10). Both flange and web would have core and separate action portions. The respective torsion constants would then be expressed as

$$(K)_F = (K_c)_F + (K_s)_F = \frac{1}{3} b T_F^3 + \frac{2}{3} n c t_F^3 + \frac{2}{3} m t_A^3 \qquad (24)$$

$$(K)_W = (K_c)_W + (K_s)_W = \frac{1}{3} e T_W^3 + \frac{4}{3} f t_A^3 + \frac{1}{3} a t_W^3 \qquad (25)$$

The torsional shear torque T_S was then expressed in terms of its component parts, $(T_S)_W$ for the web and $(T_S)_F$ for each flange as follows:

$$T_S = (T_S)_W + 2(T_S)_F = (K)_W G(\theta)_W + 2(K)_F G(\theta)_F \qquad (26)$$

When the measured values of the unit angle of twist (θ) are introduced, the torsional shear torque can be evaluated at corresponding points. Then the restraint of warping torque T_W can be computed by taking the difference between the applied torque (T) and T_S. The flange shear V_F is obtained by dividing T_W by the effective height h_e. The flange bending moment curve M_F is based on the fundamental relationship that the change in bending moment between any two points is equal to the area under the shear curve between the same two points. The flange bending stress follows from the flexure formula. The sequence of steps taken is shown diagramatically in Fig. (12).

When plotted, the measured values of the unit angle of twist for both flange and web revealed a gradual but decided drop near the free end. Considering the end supports used in the actual tests, it was assumed that the (θ) and consequently the T_S curves both converge to zero as shown by the dotted lines.

The area under the V_F curve between substations was computed assuming straight line segments. Starting at the free end where the bending moment was assumed to be zero, the flange bending moment at the successive stations was computed as the cumulative sum of the shear curve areas.

This procedure was applied to each of the built-up plate girders tested. The resulting flange bending stress variation was plotted as Comp. "H." An inspection of Fig. (11) for T5-R points up the decided improvement in the check against measured stresses as compared to the Timoshenko solution, even to the unexpected hump near the free end. Similar results were obtained for the other built-up specimens.

This study brought out the fact that the readjustment in the division of torque near the free end and the differential angular distortions of web and flange affect the stress distribution throughout the span. The division of torsion constant between flange and web previously assumed was found to give reasonable results and hence was adopted for use in the subsequent computations.

Application of Goodier-Barton Theory to Plate Girders

The "finite beam" solution was applied to the built-up sections under shaft torque with restraint of warping. After the section properties were converted into the specified dimensionless constants, the roots of the characteristic equation were determined. The arbitrary constants were computed from the boundary conditions. The resulting solution for the angular distortions was used to compute the stresses. For all specimens with relatively high web to flange thickness ratio, the roots turned out to be real. Very little difference could be found between the Goodier-Barton and Timoshenko solutions.

However, for the flange-heavy specimen T5-R, the roots were complex. The "finite beam" solution was carried out as outlined by Goodier and Barton but with the following modifications:

1. In Equation (21) for the angle of twist, the first arbitrary constant A_2 was considered as a complex number rather than just a real number.
2. The three equations based on boundary conditions required to evaluate the arbitrary constants were set up in terms of all the real and imaginary parts discarded on the basis of physical considerations.

The "finite beam" solution for the flange bending stress in specimen T5-R is shown in Fig. (11) as curve "J." The improvement over the Timoshenko solution in the check against measured values is quite marked.

On the basis of this limited investigation it appears that the Goodier-Barton theory should be used in those cases where the non-uniform torsion stresses are important or where the cross-section is of such proportions that the web deformation effect is accentuated.

Variable Torsion Constant

For purposes of checking the theory of non-uniform torsion, it was desirable to keep the specimens as simple as possible. Consequently the cross-section as well as the spacing of connectors was kept constant. However, a plate girder with variable length cover plates will present a case of a specimen with different values of torsional and lateral bending rigidities in each segment.

A plate girder with variable torsion constant (K) with or without a corresponding change in I_y can be solved by use of an extension of the Timoshenko theory. In Modification No. 2 a cantilevered girder with $(K)_1$ and $(I_y)_1$ effective over a given portion and $(K)_2$ and $(I_y)_2$ effective over the balance of the span was subjected to a shaft torque (T) at the free end as sketched in Fig. (13).

The equations for the angle of twist (Ψ), the unit angle of twist (θ) and the flange bending moment M_F are written for each segment in terms of six arbitrary constants. These are evaluated by utilizing the following six consistent boundary conditions. At the fixed end both (Ψ) and (θ) are equal to zero. M_F is zero at the free end. At the junction point where the section properties change, the values of (Ψ), (θ) and M_F from the left and right segments must be equal due to physical considerations. Substitution of the arbitrary constants leads to the expressions which will give the desired angular and stress function. A characteristic cusp is developed in both the (θ) and M_F curves.

The foregoing procedure was applied to several specimens to establish the effect of a reduction in (K) at the ends. It was found that if for any reason, the effective (K) is reduced in any portion, the result will be an increase in the angular distortion and the flange bending stress throughout the span. This effect was noticeable in the built-up specimens where the connectors were not carried out clear to the free end.

CONCLUSIONS

In general the experimental phase of this investigation progressed according to plan. The loading and support arrangements used in the laboratory tests gave good service. The instrumentation proved to be adequate. The use of a vertical bar braced against the flange sides to measure the angular twist of the web by-passed the possible complications due to web deformation.

On the basis of the observed behavior and the analytical study of results, the following conclusions were reached:

1. The test of the rolled beam added further confirmation to the contention that the Timoshenko solution will adequately predict the magnitude of stresses developed in solid sections of the usual dimensions under non-uniform torsion.
2. The Timoshenko solution was also found to be applicable to plate girders with relatively high ratio of web to flange thickness.
3. The Timoshenko theory, however, was found to err appreciably on the low side when used to calculate the flange bending stresses in a plate girder with relatively heavy flanges.
4. A procedure was developed to check the validity of a suggested grouping of section properties for plate girders. Based on the compatibility of test data the measured web and flange distortions were used to compute the flange bending stresses which compared well with measured values.
5. The "finite beam" solution of the Goodier-Barton theory predicted with reasonable accuracy the effect of web deformation on the angle of twist of cantilevered girder specimens subjected to shaft torque.
6. The "finite beam" solution gave flange bending stresses which checked well with measured values in all of the plate girder specimens tested.
7. Any reduction in the effective torsion constant along the span will result in an increase in stresses and angular distortions.
8. In those cases where the stresses and distortions must be determined with precision the Goodier-Barton theory is indicated. For design purposes the much simpler Timoshenko solution may be preferred.

APPENDIX A BIBLIOGRAPHY

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ACKNOWLEDGMENT

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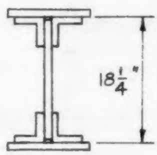
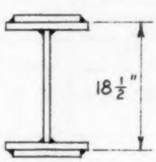
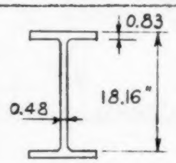
TABLE 1 SUMMARY OF TEST SPECIMEN

Mark	Sketch	No. of Cover Plates per Flg.	Fabrication	Test $(K)_A$ in^4	Separate action $(K_S)_D$ in^4
T1-R		None	Riveted	1.86	1.56
T2-R		1	Riveted	6.61	1.87
T3-B		1	Bolted	7.42	1.87
T5-R		3	Riveted	17.40	2.43
T7a-W		None	Welded	2.35	-
T7b-W		1	Welded	6.35	2.35
T9-WF			Rolled 18WF-77	3.83	-

MATERIAL

Riveted & Bolted Specimen		Welded Specimen	
Web Plate	18 x 3/8	Web Plate	17-1/4 x 3/8
Flange Plate	4 x 3-1/2 x 1/2	Flange Plate	9-1/2 x 5/8
Cover Plate	9-1/4 x 3/8	Cover Plate	8-1/2 x 1/2

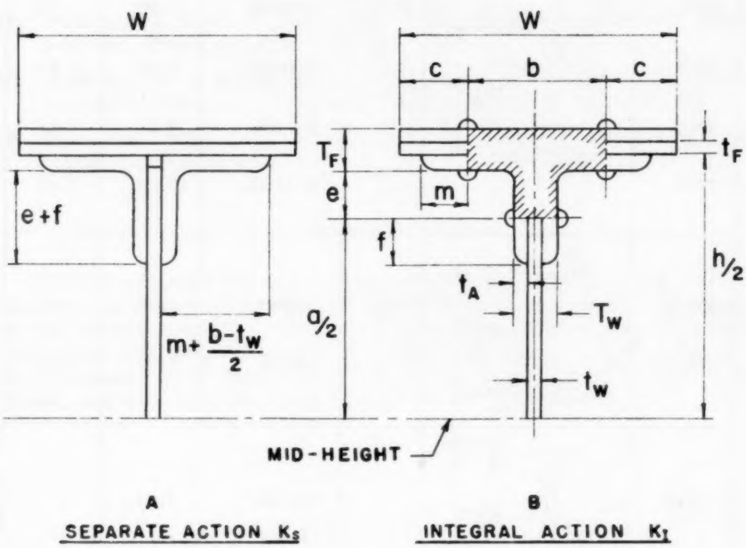


FIG. I

TORSION CONSTANT K

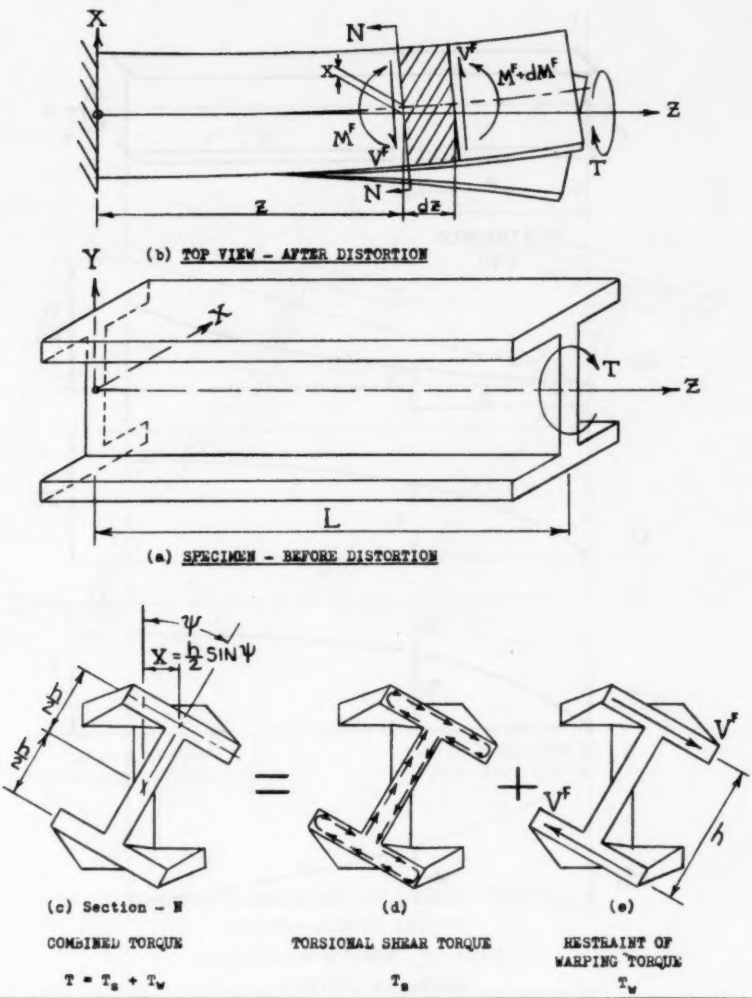


Fig. 2 NON-UNIFORM TORSION OF WF BEAM - SHAFT LOADING

TIMOSHENKO THEORY - WITHOUT WEB DEFORMATION

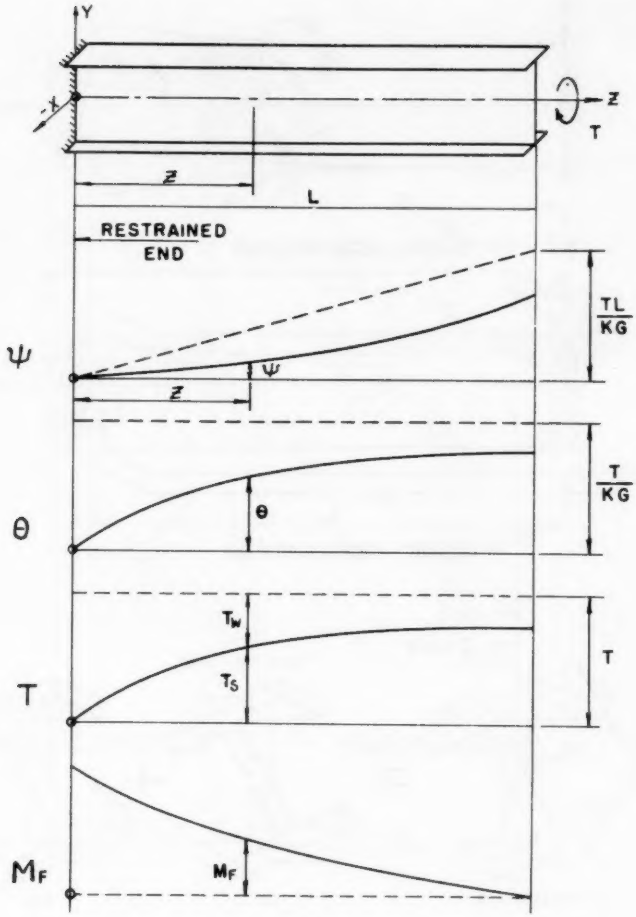


FIG. 3

SUMMARY CURVES

TIMOSHENKO SOLUTION NON-UNIFORM TORSION OF WF BEAM

——— NON-UNIFORM TORSION

- - - - UNIFORM TORSION

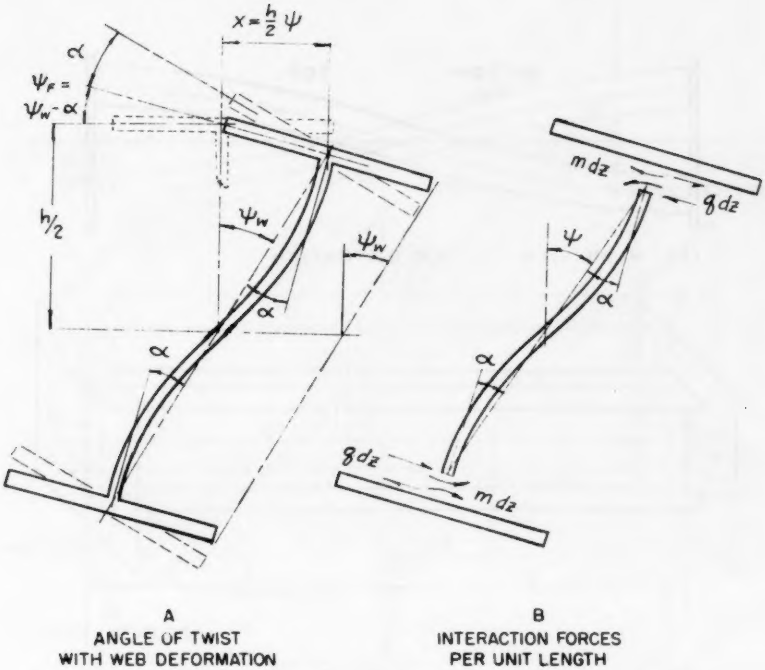
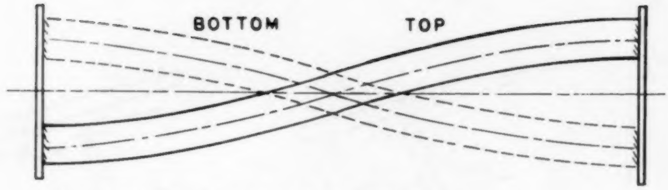
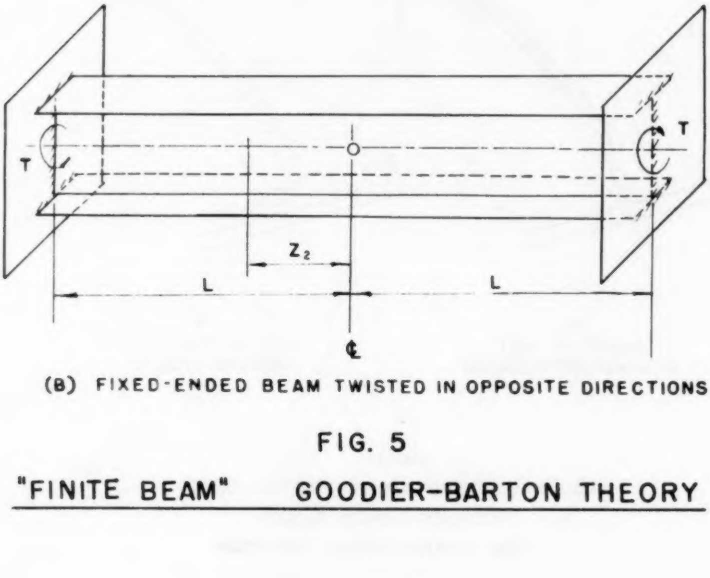


FIG. 4
ANGULAR DISTORTION AND INTERACTION FORCES
GOODIER-BARTON THEORY
NON-UNIFORM TORSION WF BEAM



(A) PLAN VIEW - AFTER DISTORTION



(B) FIXED-ENDED BEAM TWISTED IN OPPOSITE DIRECTIONS

FIG. 5

"FINITE BEAM" GOODIER-BARTON THEORY

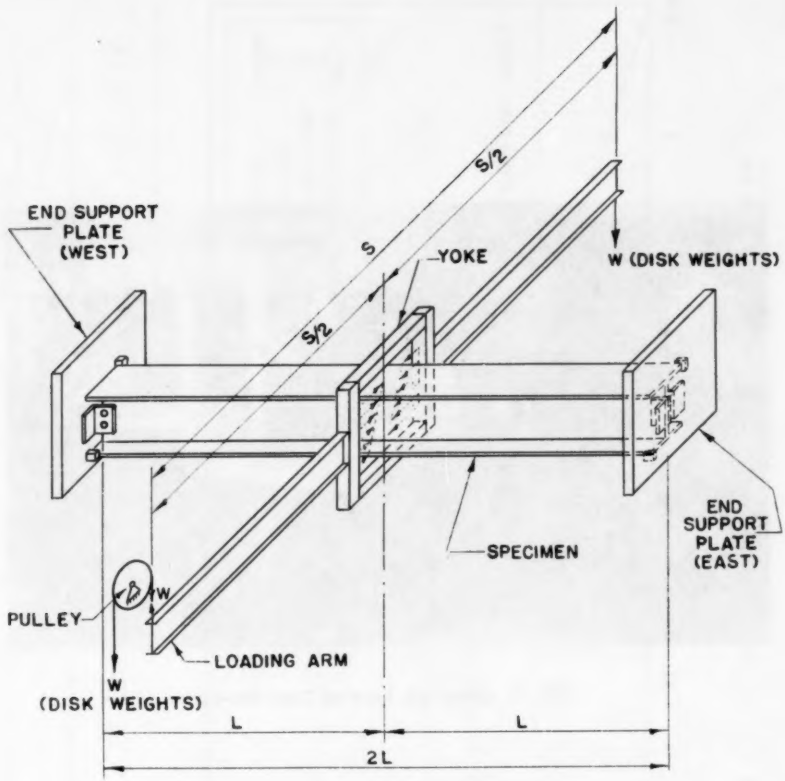


FIG. 6

TEST SETUP

APPLICATION OF TORQUE
SCHEMATIC DRAWING

FRITZ ENGINEERING LABORATORY PROJECT NO. 215A.
"NON-UNIFORM TORSION OF PLATE GIRDERS"

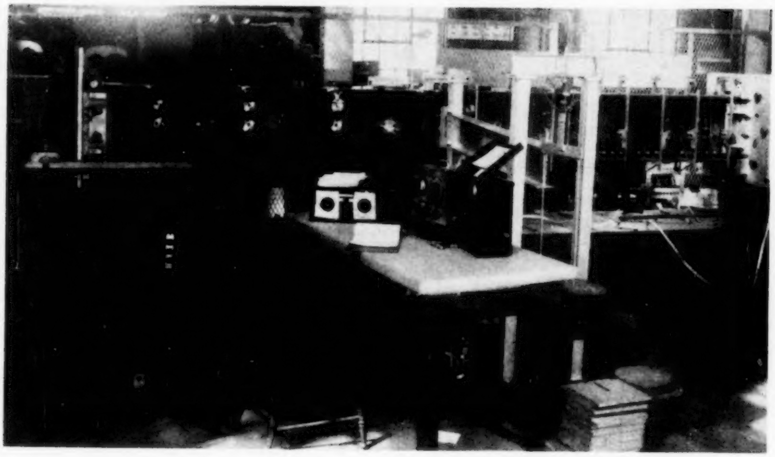


FIG. 7. Over-all View of Test Set-up

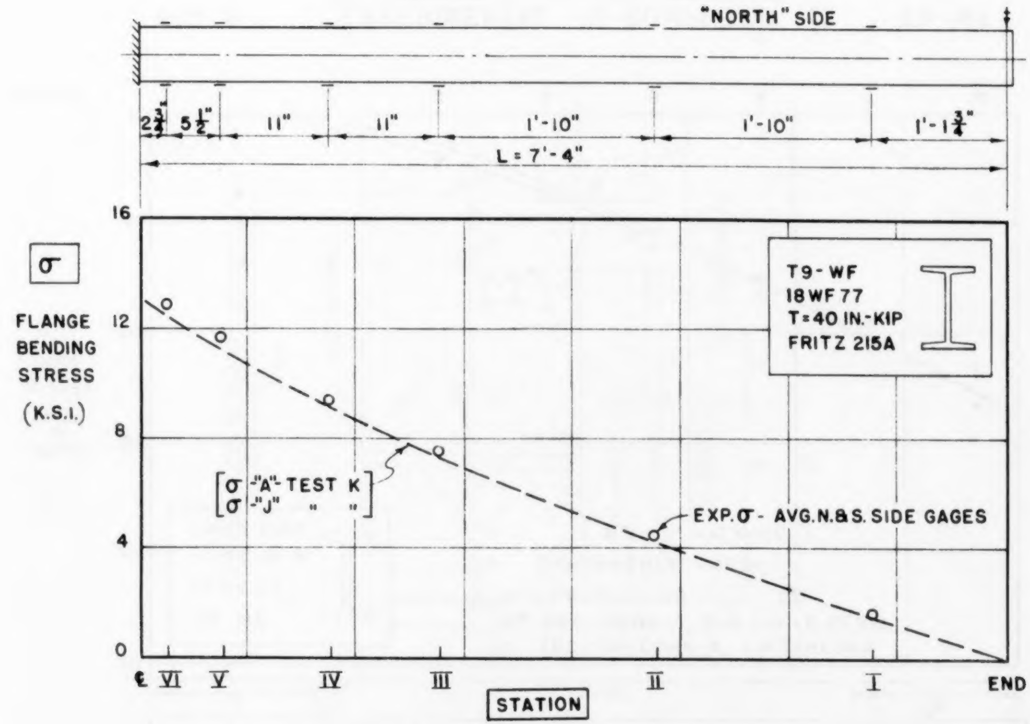


FIG. 8 EXPERIMENTAL VS. COMPUTED σ T9-WF

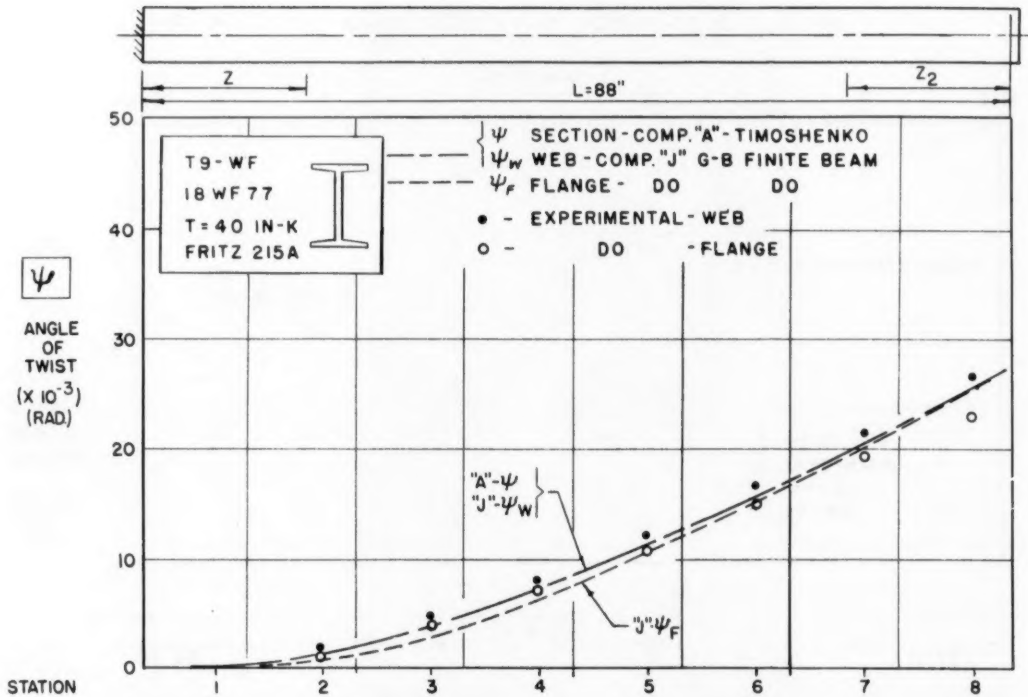


FIG. 9 EXPERIMENTAL VS COMPUTED ψ T9-WF

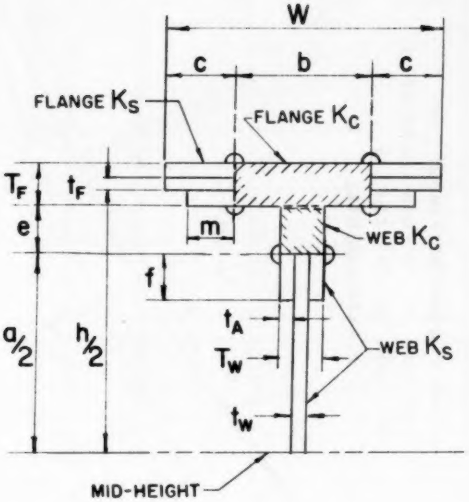


FIG. 10

DIVISION OF TORSION CONSTANT K

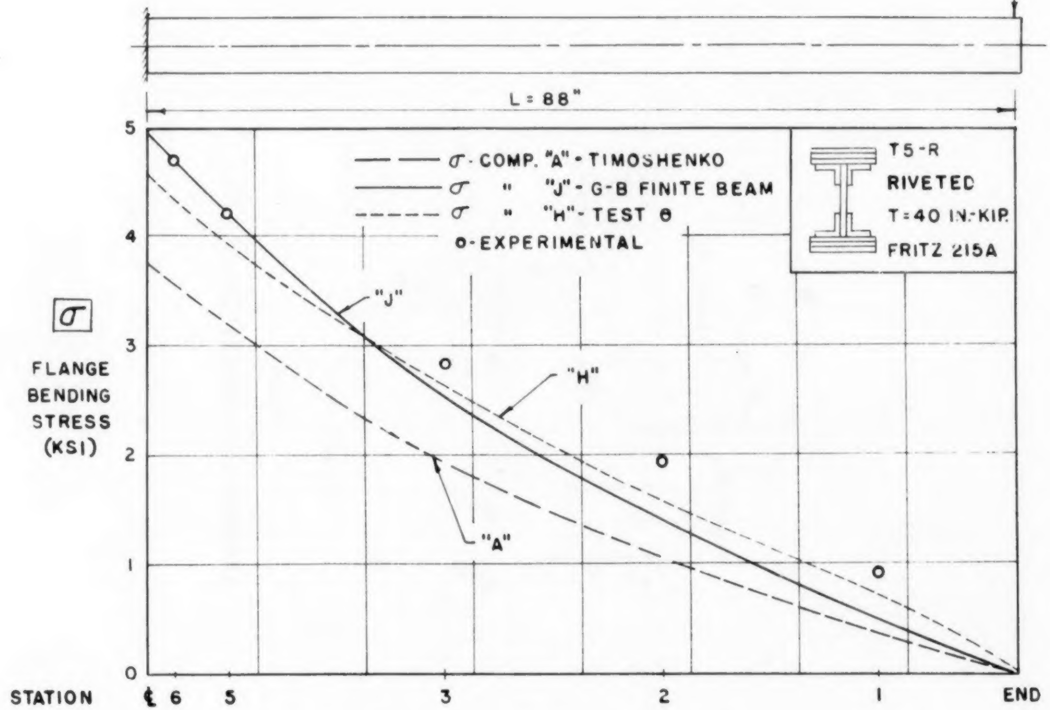


FIG. 11 EXPERIMENTAL VS COMPUTED σ T5-R

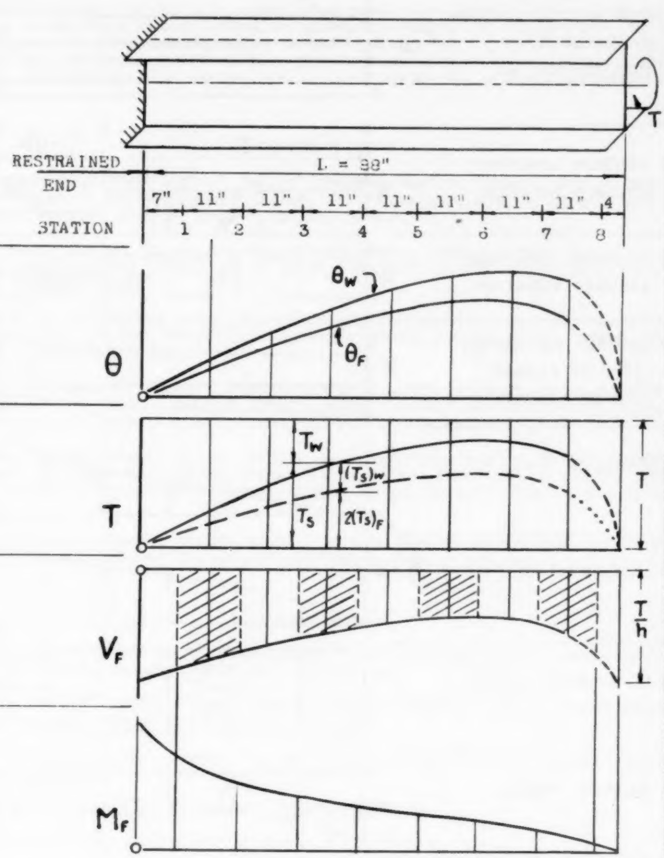


FIG. 12

NON-UNIFORM TORSION SHAFT LOADING

COMPUTATION "H" MODIFICATION NO. 1

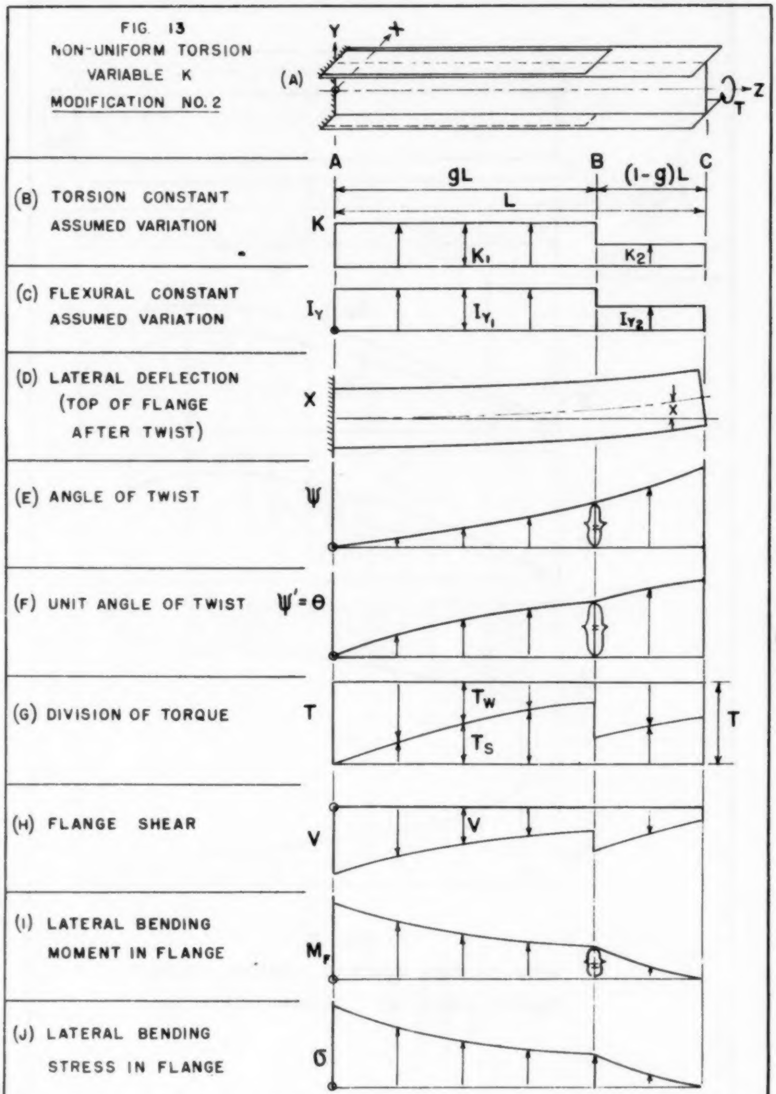


FIG. 13
NON-UNIFORM TORSION
VARIABLE K
MODIFICATION NO. 2

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a. Beginning with "Proceedings-Separate No. 200," published in July, 1953, the papers were printed by the photo-offset method.
b. Presented at the Miami Beach (Fla.) Convention of the Society in June, 1953.
c. Presented at the New York (N.Y.) Convention of the Society in October, 1953.
d. Beginning with "Proceedings-Separate No. 290," published in October, 1953, an automatic distribution of papers was inaugurated, as outlined in "Civil Engineering," June, 1953, page 66.
e. Discussion of several papers, grouped by divisions.
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